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NONLOCALITY IMPROVES DEUTSCH ALGORITHM

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We show that the Bell inequalities lead to a new type of linear-optical Deutsch algorithms. We have considered the use of entangled photon pairs to determine probabilistically two unknown functions. The usual Deutsch algorithm determines one unknown function and exhibits a two to one speed up in a certain computation on a quantum computer rather than on a classical computer. We found that the violation of Bell locality in the Hilbert space formalism of quantum theory predicts that the proposed *probabilistic* Deutsch algorithm for computing two unknown functions exhibits at least a $2\sqrt{2}(\simeq 2.83)$ to one speed up.

Keywords: Deutsch algorithm; Bell inequality; quantum computing.

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1. Introduction

Quantum information processing and quantum computing have attracted much interest in the science community because of the novel usage of quantum mechanics in technological applications. Many ideas of quantum processors and computers were experimented in many architectures of physical systems, including ion traps, neutral atoms, nuclear spins in magnetic resonance, semiconductor quantum dots, and super-conducting resonators.[1–5] Still the progress has been limited to a few qubit operations and the performance is far from being practical. The ability of quantum computers in outperforming their classical counterparts has not been demonstrated.

In many physical systems, the preparation and change of quantum states, or how to prepare entanglements and how to maintain coherence, is a more difficult task than how to wire logics quantum-mechanically. However, in quantum computing with linear optical systems, it is relatively easy to deal with entanglement and decoherence. For this reason, linear optical quantum computings or the linear interactions of photons with matters are often adopted for the implementation of N-qubit quantum algorithms.[6,7] Especially entanglement is an important aspect that a quantum mechanical device can have, and the quantum information carried by an entangled state like Einstein, Podolsky, and Rosen (EPR) state overcomes some of the limitations of classical information used in communication and cryptography.[8–13] Recently, there have been several attempts to use single-photon two-qubit states for quantum computing. Oliveira *et al.* implemented the Deutsch algorithm with polarization and transverse spatial modes of the electromagnetic field as qubits.[14] Single-photon Bell states were prepared and measured by Kim.[15] Also the decoherence-free implementation of Deutsch algorithm using such single-photon two logical qubits.[16] Although such single-photon two-qubit implementations are not scalable, the quantum gates necessary for information processing can be implemented deterministically using only linear optical elements.

Often a demonstration of quantum algorithm, which is performed with possibly mixed quantum states, is presented without a proper mathematical theory for the analysis of experimental data, or sometimes with a rather complicated quantum tomographical state analysis. However, if the output state of an experiment under study should be an entangled state, we can choose to use Bell inequalities.[17] It is a sufficient condition to demonstrate a negation of Bell locality and the detection of entanglement. We can consider the following question. Is there a relationship between the negation of Bell locality and the performance of such quantum algorithm that can be implemented by a single photon? Interestingly the answer is yes.

In this paper we have devised an experimental scheme to obtain simultaneous and nonlocal answers from Deutsch problem with two unknown functions. Especially we elaborate the use of *entanglement* in processing this quantum algorithm. The advantage of using an EPR pair of photons (two-photon two-qubit states) in quantum computing algorithm is analyzed with Bell inequalities in quantum theory. We show that a set of answers of the given Deutsch problem, with two unknown functions, statistically shows a violation of Bell locality in the Hilbert space formalism of quantum theory. It turns out that the negation of Bell locality exhibits a $2\sqrt{2}$ to one speed up at least. We, thus, observe a highly nonlocal effect and it leads the entangled answers in a network of pair quantum computers to provide enhanced information compared with its classical counterpart. An important note here is that there must be probabilistic errors in answers which appear due to the imperfection of the photon detection and defects in optical devices. Thus it is necessary to take into account many measurements and what we can do is only to analyze probabilistically in real experimental situation. By applying the maximum likelihood principle, the answer to the Deutsch problem is estimated probabilistically.

In the following sections, we introduce a method of linear optical quantum computing of Deutsch problem. In Sec. 2, the method that utilizes two-photon two-qubit entanglement is discussed. This method allows statistical analysis of the average value of Bell operator. In order to overcome the imperfection of the photon detection and possible defects in optical devices, the fidelity[20] of the method is analyzed. During the analysis of Bell operator or the fidelity to EPR state in Sec. 3, the separable states are distinguished from the entangled states. It is well-known that the fidelity which is larger than $1/\sqrt{2}$ to EPR state is a sufficient condition for a negation of Bell locality in the Hilbert space formalism of quantum theory. We can say that our Deutsch scheme succeeds with the probability of the value of the lower bound of fidelity at least under the condition where the fidelity is larger than $1/\sqrt{2}$. A short summary and conclusion follows in Sec. 4.

2. Deutsch Algorithm with Two Unknown Functions

The Deutsch algorithm determines whether a given function, $f(x)$, on a binary number, x, is either balanced or constant, where the function is constant if the function outputs $f(0)$ and $f(1)$ are the same and balanced if not. The unknown function is defined by

$$U_f|x\rangle_y \equiv |x\rangle_{y\oplus f(x)}. \tag{1}$$

In a classical computer, the answer is obtained by calculating $f(0)$ and $f(1)$, while in a quantum computer only a single calculation with a superposition state of 0 and 1 is necessary. Hence, if $f(H) = f(V)$ then the output label should not depend on H and V, whereas if $f(H) \neq f(V)$ then the output label should depend on H and V.

As shown in Fig. 1, Deutsch algorithm may be implemented using a two-photon state. The two qubits in channels (B) and (C) represent the polarization and momentum (or optical path) states of the photon pair. The horizontally polarized state in channel (B) undergoes a Hadamard gate (**H**) and forms a superposition state of polarization states. Then, a coherent superposition state of the optical paths of the photon is created pertained to the polarization state, by a polarization beam splitter, for example. The CNOT operation is performed conditionally if a balanced function is chosen and if a constant function is chosen, an identity

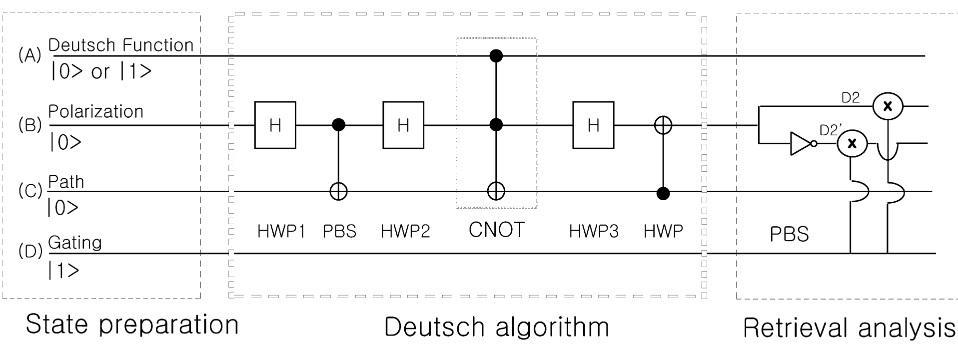

Fig. 1. The schematic of Deutsch algorithm with an entangled two-photon state. The classical channel (A) represents the type of the chosen Deutsch function, $|0\rangle$ for a balanced function and $|1\rangle$ for a constant function. The polarization state and the optical path are stored in channel (B) and (C) respectively. In channel (B), $|0\rangle$ denotes a horizontal polarization state and $|1\rangle$ denotes a vertical polarization state. In channel (C), $|0\rangle$ ($|1\rangle$) denotes the optical path denoted by 2 (2′) or $a(b)$ and toward D2 (D2′) shown in Fig. 2. The channel (D) is for the other photon of the photon pair which is used for a time correlation detection.

operation (**I**) is performed. Corresponding three qubit operation is Toffoli gate, or C-CNOT gate. The polarization state goes through another Hadamard gate and then another CNOT gate but in this case the control and target bits are exchanged. The final qubit state of the photon in the channel (D) is collapsed and detected either at the detector D2 or D2′, to reveal the type of the Deutsch function: whether the unknown function was balanced or constant. The other photon, in the vertical polarization, is used as a gate function for the coincidence measurement. This is reminiscent of Oliveira's scheme in Ref. 14, except that we discuss quantum-mechanical advantage of wiring two of those Deutsch algorithms processed by an entangled two-photon state.

We describe the experimental scheme of which is shown in Fig. 2. This setup has two Deutsch algorithms, shown in Fig. 1, in the left and right sides of the setup. We note that the two functions are not simultaneously processed but probabilistically processed one by one. Yet the statistical analysis to be described in Sec. 3 proves a processing speed-up due to quantum mechanical properties. The unknown functions A and B in the gray boxes in Fig. 2 represent the Toffoli gate in Fig. 1. The half-wave-plates labeled from HWP1 to HWP5, oriented at $\theta = 22.5°$, are Hadamard gates that change a horizontal (vertical) polarization state into a 45° (−45°) polarization state. The half-wave-plates labeled as HWP in Fig. 2 are oriented at $\theta = 45°$ and interchange the polarization state of the photon in one optical path (b) *only* and not one in the other path. Corresponding operation is the CNOT gate with the states of the optical path as the control bits and with the polarization states as the target bits.

In our implementations, we assign the truth values 0 and 1 as $|H\rangle \leftrightarrow |0\rangle$, $|V\rangle \leftrightarrow |1\rangle$. The initial photon becomes a coherent superposition state, $(|0\rangle_2 + |1\rangle_2)/\sqrt{2}$

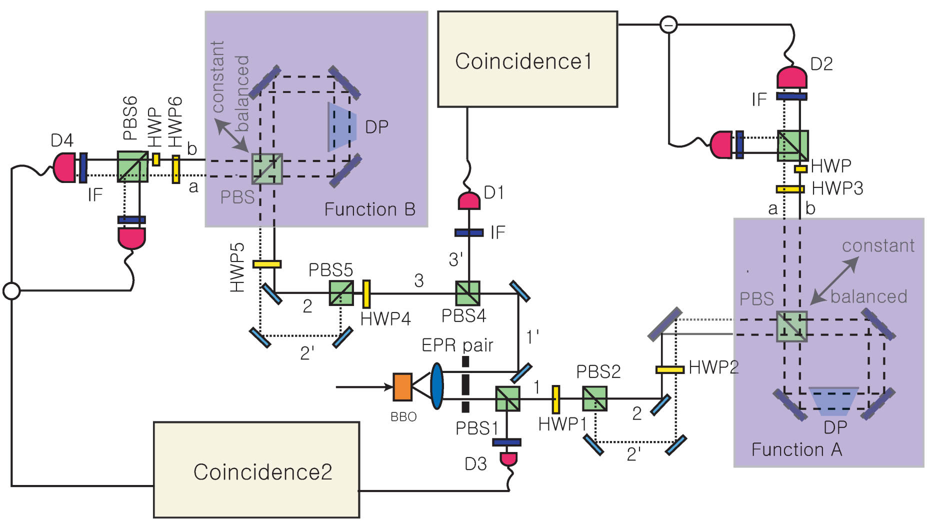

Fig. 2. The Deutsch algorithm with a two-photon entangled state.

after the first half-wave-plate (HWP1 or HWP4). Then at the polarization beam splitter (PBS2 or PBS5), the photon path is determined by the polarization state, or $(|0\rangle_2 + |1\rangle_{2'})/\sqrt{2}$. Now the coherent superposition state of polarization states as well as in the two different optical paths is prepared. Once again, the polarization qubit undergoes a Hadamard gate, at HWP2 (or HWP5), before the photon enters the Toffoli gate, or the conditional-CNOT gate constructed with the optics in the gray boxes in Fig. 2. HWP3 (or HWP6) performs a Hadamard operation and HWP performs a CNOT operation with the polarization state as its target bit. Then, the photon with the horizontal polarization reaches D2 and that with the vertical polarization reaches D2′.

In qubit notations of $|px\rangle$ where p and x are the polarization qubit and the optical path qubit, respectively, the described process in Fig. 1 becomes

$$\mathbf{U_{CN'}} \mathbf{U_{H_1}} \left\{ \begin{matrix} \mathbf{U_{CN}} \\ I \end{matrix} \right\} \mathbf{U_{H_1}} \mathbf{U_{CN}} \mathbf{U_{H_1}} |00\rangle = \left\{ \begin{matrix} |1+\rangle \\ |0+\rangle \end{matrix} \right\}, \tag{2}$$

where $\mathbf{U_{H_1}} = \mathbf{H} \otimes \mathbf{I}$, $\mathbf{U_{CN}}$, and $\mathbf{U_{CN'}}$ are the Hadamard gate acting on the polarization qubit, a CNOT gate with polarization control bit, and a CNOT gate with polarization target bit, respectively. $|\pm\rangle$ denotes $(|0\rangle \pm |1\rangle)\sqrt{2}$. The top and bottom row of the parenthesis represents the balanced and constant Deutsch function cases, respectively.

The reason we use other photon, in channel (D), for the coincident measurement is to determine the lower bound of the success probability of the Deutsch algorithm and to see a violation of Bell locality in the Hilbert space formalism of quantum theory. In this section, we consider an ideal case, i.e. there is not any experimental noise to simplify the discussion. However, in real experimental situations, we have to take error answers into account due to experimental imperfections. Thus, many runs of experiments are evidently necessary. Hence, we shall discuss a method using Bell operators to analyze experimental data. The method of such analysis will be presented in Sec. 3.

When two Deutsch functions, A and B, are processed together, there are 8 different channels, 4 for the function A and 4 for the function B. In the following subsections, we provide a step-by-step validation of the process described in Eq. (2) and the combined result of two of such processes. We first start with the initial state of photons, an EPR photon pair, from the BBO crystal, as:

$$\frac{1}{\sqrt{2}}(|0\rangle_1 |1\rangle_{1'} - |1\rangle_1 |0\rangle_{1'}), \tag{3}$$

where the photon paths 1 and 1′ are shown in Fig. 2. Now we follow the time evolution of each of photons.

2.1. *Deutsch algorithm with a function A*

Assume that the case where the state $|0\rangle_1 |1\rangle_{1'}$ is used to determine the function A shown on the left side of Fig. 2. The unknown function A is determined as constant or balanced. The vertically polarized photon along the path $1'$ is measured as the digital 1 at the detector D1, or $|1\rangle_{1'} \rightarrow |1\rangle_{D1}$, and the horizontally polarized photon along the path 1 becomes $|1\rangle_1 \rightarrow (|0\rangle_1 + |1\rangle_1)\sqrt{2}$, hence the photon state becomes

$$\mathbf{H}|0\rangle_1|1\rangle_{1'} = |+\rangle_1|1\rangle_{D1}. \tag{4}$$

The vertically polarized photon is detected by the detector $D1$ as

$$\sigma_z^{D1} = -1, \tag{5}$$

and, therefore, the state of the photons $|+\rangle_1|1\rangle_{D1}$ is projected into $|+\rangle_1$. The polarizing beam splitter (PBS2) changes each of states as $|0\rangle_1 \rightarrow |0\rangle_2,\ |1\rangle_1 \rightarrow |1\rangle_{2'}$. Hence we have the coherently superposed states of a photon in two paths 2 and $2'$ as,

$$|+\rangle_1 \rightarrow \frac{1}{\sqrt{2}}(|0\rangle_2 + |1\rangle_{2'}), \tag{6}$$

or in the two-qubit notation $|px\rangle$, $(|00\rangle + |11\rangle)/\sqrt{2}$. Then the half-wave plate (HWP2), with the optic axis oriented at $\theta = 22.5°$, performs Hadamard operations on the polarization qubits, or $\mathbf{H} \otimes \mathbf{I}(|00\rangle + |11\rangle)/\sqrt{2}$, and we get

$$\frac{1}{\sqrt{2}}(|+0\rangle + |-1\rangle) = \frac{1}{\sqrt{2}}(|0+\rangle + |1-\rangle). \tag{7}$$

A dove prism (DP) is the most important part in implementing the given functions. The output polarization state of the dove prism rotates at twice the angular rate of the rotation of the prism itself. So, if a dove prism inclines at 45° about a vertical line, the image rotates at 90°. Now we consider how a CNOT gate (cf. Ref. 21) is implemented with the space and polarization degrees of freedom of a

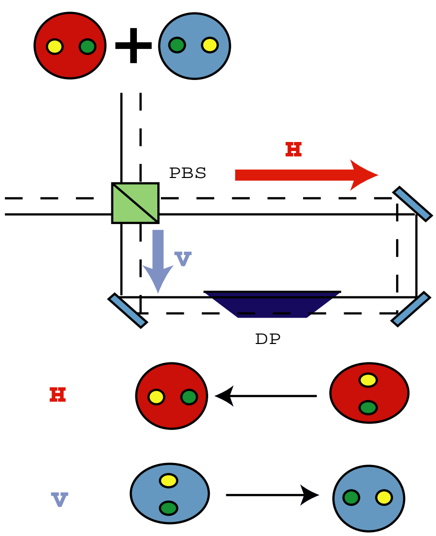

Fig. 3. An example of polarization-CNOT gates.

photon. As shown in Fig. 3, a horizontally polarized photon is transmitted through the polarization beam splitter. But a vertically polarized photon is reflected at the polarization beam splitter. They propagate along different ways. The angle between a vertical line and the axis of the dove prism is $45°$ in the case of a horizontally polarized photon, but $-45°$ in the other case. So this configuration implements such changes of photon paths: $2 \to a$, $2' \to b$ when the polarization is horizontal. $2 \to b$, $2' \to a$ when the polarization is vertical. Here a and b are the labels for the paths toward detectors. Output photons labeled by a and b are detected by D2 and D2′.

2.1.1. *Balanced function case*

In the balanced function case, the changes of the polarization states of the photon due to the dove prism are:

$$\begin{aligned} &|0\rangle_2 \to |0\rangle_a, \quad |0\rangle_{2'} \to |0\rangle_b, \\ &|1\rangle_2 \to |1\rangle_b, \quad |1\rangle_{2'} \to |1\rangle_a, \end{aligned} \tag{8}$$

of which the operation is a controlled NOT with the polarization and path bits as the control and target bits. We denote the 2-a path digital 0 and 2′-b digital 1. Thus, after the dove prism, the qubits (7) becomes $(|0+\rangle - |1-\rangle)/\sqrt{2}$. After HWP3, we have $(|01\rangle + |10\rangle)/\sqrt{2}$ and after HWP$(\theta = 45°)$, we get $(|10\rangle + |11\rangle)/\sqrt{2}$. Hence, we detect only vertically polarized photons in $D2$. This implies

$$(\sigma_z^a = \sigma_z^b =)\sigma_z^{D2} = -1. \tag{9}$$

Hence from Eqs. (5) and (9), the value of observable $\sigma_z^{D1}\sigma_z^{D2}$ should be $+1$. This is one of the success result of a single run of the experiment, i.e.

$$\sigma_z^{D1}\sigma_z^{D2} = +1. \tag{10}$$

2.1.2. *Constant function case*

After the dove prism, polarized photon states change as follows

$$\begin{aligned} &|0\rangle_2 \to |0\rangle_a, \quad |0\rangle_{2'} \to |0\rangle_b, \\ &|1\rangle_2 \to |1\rangle_a, \quad |1\rangle_{2'} \to |1\rangle_b. \end{aligned} \tag{11}$$

Similarly, we get after HWP3 and HWP$(\theta = 45°)$ $(|00\rangle + |01\rangle)/\sqrt{2}$. Hence, we detect only horizontally polarized photons in $D2$. This implies

$$(\sigma_z^a = \sigma_z^b =)\sigma_z^{D2} = +1. \tag{12}$$

Hence from Eqs. (5) and (12), the value of observable $\sigma_z^{D1}\sigma_z^{D2}$ should be -1. This is one of the success result of a single run of the experiment, i.e.

$$\sigma_z^{D1}\sigma_z^{D2} = -1. \tag{13}$$

Thereby, we can determine whether a given function (A) is constant or balanced with utilizing the state $|0\rangle_1|1\rangle_{1'}$.

2.2. *Deutsch algorithm with a function B*

Similarly we can assume that the case where the state $|1\rangle_1|0\rangle_{1'}$ is contributed to our Deutsch algorithm. In this case, the unknown function (B) is determined in constant one or balanced one. At the detector $D3$, the photon state becomes $(|0\rangle_1 + |1\rangle_1)|1\rangle_{D3}/\sqrt{2}$. So a vertically polarized photon is detected by the $D3$ detector. That is, $\sigma_z^{D3} = -1$. Therefore in the balanced function case, the changes of the polarization states of the photons due to the dove prisms, HWP6 and the last HWP ($\theta = 45°$) are: $(|10\rangle + |11\rangle)/\sqrt{2}$. Hence, we detect only vertically polarized photons in $D4$. This implies $(\sigma_z^a = \sigma_z^b =)\sigma_z^{D4} = -1$. Hence the value of observable $\sigma_z^{D3}\sigma_z^{D4}$ should be $+1$. This is one result of a single run of the experiment, i.e.

$$\sigma_z^{D3}\sigma_z^{D4} = +1. \tag{14}$$

In the constant function case, similarly we get $(\sigma_z^a = \sigma_z^b =)\sigma_z^{D4} = +1$. Hence the value of observable $\sigma_z^{D3}\sigma_z^{D4}$ should be -1. This is one of the success result of a single run of the experiment, i.e.

$$\sigma_z^{D3}\sigma_z^{D4} = -1. \tag{15}$$

Thereby, we can determine whether a given function (B) is constant or balanced with utilizing the state $|1\rangle_1|0\rangle_{1'}$.

Thus, we can determine whether either a given function (A) or (B) is constant or balanced with utilizing EPR entanglement. Clearly, many EPR experiments evaluate two functions simultaneously, i.e. Deutsch algorithm exhibiting a four to one speed up. In the next section, we assume the existence of experimental imperfections and we present the method to determine the lower bound of the success probability of our scheme presented by Fig. 2. Especially, a violation of Bell locality in the Hilbert space formalism of quantum theory ensure that the success probability is larger than $1/\sqrt{2}$.

3. Bell Operator Analysis

In the previous sections, we have assumed that the initial state is a two-photon entangled state $|\Psi\rangle = (|V\rangle_z|H\rangle_z - |H\rangle_z|V\rangle_z)/\sqrt{2}$. We now insert a polarizer oriented at 45° and a HWP ($\lambda/2$ plate) in front of each detector. See Fig. 4. This allows the measurement of polarized photon states described in polarized basis x. That is, one can measure an observable σ_x in this way. Due to the feature of the initial state, the same situation occurs in the ideal case. The situation is as follows. One can see

$$\begin{aligned} |H\rangle_x &= \frac{|H\rangle_z + |V\rangle_z}{\sqrt{2}}, \\ |V\rangle_x &= \frac{|H\rangle_z - |V\rangle_z}{\sqrt{2}}. \end{aligned} \tag{16}$$

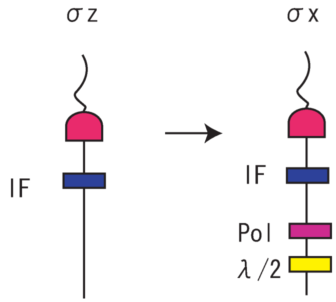

Fig. 4. A setup of measurement of polarization in z basis and in x basis.

Let us rewrite the initial state $|\Psi\rangle$ using x polarization basis. We have $|\Psi\rangle = (|V\rangle_x|H\rangle_x - |H\rangle_x|V\rangle_x)/\sqrt{2}$. This implies that the scheme mentioned in the preceding section works in the same way. However, we have to take the imperfection of the photon detection and defects in optical device into account.

Here we introduce Bell operators:

$$\begin{aligned} B_A &= \frac{1}{\sqrt{2}}(\sigma_x^{D1}\sigma_x^{D2} + \sigma_z^{D1}\sigma_z^{D2}) \\ B_B &= \frac{1}{\sqrt{2}}(\sigma_x^{D3}\sigma_x^{D4} + \sigma_z^{D3}\sigma_z^{D4}). \end{aligned} \tag{17}$$

First of all, we check if both of the following Bell inequalities[17] are violated:

$$|\langle B_A\rangle_{\mathrm{avg}}| \leq 1, \quad |\langle B_B\rangle_{\mathrm{avg}}| \leq 1. \tag{18}$$

When both of the Bell inequalities are violated, we can ensure that the success probability of our Deutsch algorithm is larger than $1/\sqrt{2}$ in the experiment as shown below.

We note here that if experimental error exists, one could misjudge the unknown function. For instance, it is possible that actually observed data says that the unknown function is constant even though the unknown function is in fact balanced. Such a wrong case occurs when experimental error is larger than a half. Nevertheless, our analysis rules out such a wrong case since a violation of two Bell inequalities ensures the success probability of our scheme is larger than $1/\sqrt{2}$.

In Table 1, we summarize the relationship between a violation of Bell inequalities and the two types of functions (A) and (B).

Table 1. The rationship between the the violation of Bell inequalities and the two kinds of functions (A) and (B).

	A	B
Balanced	$\langle B_A\rangle_{\mathrm{avg}} > 1$	$\langle B_B\rangle_{\mathrm{avg}} > 1$
Constant	$\langle B_A\rangle_{\mathrm{avg}} < -1$	$\langle B_B\rangle_{\mathrm{avg}} < -1$

The situation is as follows. First, we consider the case in which the unknown function (A) is balanced. The fidelity to $(|H\rangle_{D1}|H\rangle_{D2} + |V\rangle_{D1}|V\rangle_{D2})/\sqrt{2}$ in some quantum state ρ (the success probability) is bounded as[20]

$$\left\langle \frac{B_A}{\sqrt{2}} \right\rangle \leq f_b^A \leq \frac{\langle B_A/\sqrt{2}\rangle + 1}{2}. \tag{19}$$

In an ideal case, we have $\langle B_A \rangle = \sqrt{2}$. In the presence of experimental noise, we have

$$\langle B_A \rangle = \frac{1}{\sqrt{2}}(\langle \sigma_x^{D1}\sigma_x^{D2}\rangle_{\text{avg}} + \langle \sigma_z^{D1}\sigma_z^{D2}\rangle_{\text{avg}}). \tag{20}$$

Hence, we can determine the range of the value of the success probability f_b^A in the presence of experimental noise. Thus, a violation of Bell inequalities implies the success probability f_b^A is larger than $1/\sqrt{2}$ at least. We can analyze the case where the unknown function (B) is balanced in a similar way.

Next, we consider the case where the unknown function (A) is constant. The fidelity to $(|H\rangle_{D1}|V\rangle_{D2} - |V\rangle_{D1}|H\rangle_{D2})/\sqrt{2}$ in some quantum state ρ (the success probability) is bounded as

$$-\left\langle \frac{B_A}{\sqrt{2}} \right\rangle \leq f_c^A \leq \frac{-\langle B_A/\sqrt{2}\rangle + 1}{2}. \tag{21}$$

In the ideal case, we have $\langle B_A \rangle = -\sqrt{2}$. In the presence of the experimental noise, we have

$$\langle B_A \rangle = -\frac{1}{\sqrt{2}}(\langle \sigma_x^{D1}\sigma_x^{D2}\rangle_{\text{avg}} + \langle \sigma_z^{D1}\sigma_z^{D2}\rangle_{\text{avg}}). \tag{22}$$

Hence, we can determine the range of the value of the success probability f_c^A in the presence of experimental noise. Thus, a violation of Bell inequalities implies the success probability f_c^A is larger than $1/\sqrt{2}$ at least. We can analyze the case where the unknown function (B) is constant in a similar way.

We have two functions (A) and (B). The global success probability of our Deutsch algorithm is given by

$$P_{\text{success}} = \frac{f_{k_1}^A + f_{k_2}^B}{2} \quad (k_1, k_2 \in \{b, c\}). \tag{23}$$

Clearly, this value P_{success} is equal to the global probability with which a perfectly entangled state is detected.

As an example, suppose that the case where conditions $\langle B_A \rangle_{\text{avg}} > 1$ and $\langle B_B \rangle_{\text{avg}} < -1$ are met. We can know that the unknown function (A) is balanced and (B) is constant. In this case, our Deutsch scheme presented in Fig. 2 succeeds with the probability of the value $(\langle B_A/\sqrt{2}\rangle_{\text{avg}} + \langle -B_B/\sqrt{2}\rangle_{\text{avg}})/2$ at least. This value is equal to the lower bound of the probability with which a perfect entangled state is detected, i.e. the lower bound of the global success probability of our Deutsch algorithm. We can analyze other cases (there are four cases in fact) in the same way. So we can probabilistically determine whether each of two functions is constant or balanced simultaneously, based on a violation of two Bell inequalities and the evaluation of the success fidelity with utilizing two-photon entangled state. This implies *probabilistic* Deutsch algorithm exhibiting a four to one speed up in ideal case. A violation of Bell locality in Hilbert space says probabilistic Deutsch algorithm exhibiting a $2\sqrt{2}$ to one speed up at least.

4. Summary and Conclusion

In summary, we have presented a linear-optical implementation of quantum algorithm with the use of entanglement of photon states. For the process of Deutsch algorithm, two-photon two-qubit entangled states have been considered in conjunction with a polarization-based C-NOT gate. The algorithm presented here is the only algorithm which incorporates the Deutsch algorithm with a violation of Bell inequalities, to date. A violation of Bell inequalities ensures the success of probabilistic Deutsch algorithm with two unknown functions which exhibits at least a $2\sqrt{2}$ to one speed-up probabilistically. The global nonlocal effect leads us to make a quantum computer faster than usual ones.

Acknowledgments

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