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New Method to Reveal the Conflict Between Local Realism and Quantum Mechanics

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We formulate the expectation value of the Bell-Żukowski operator acting on qubit states of a two-particle Bell experiment. By using the equivalence between a set of N copies of a two-qubit experiment and a standard two-setting Bell experiment in an entangled $2N$-particle state, we obtain an inequality, which we may call the Bell-Żukowski inequality. It determines whether the measured correlation functions of two-particle states can be modeled locally and realistically. In this Bell experiment of two particles, the conflict between local realism and quantum mechanics is discussed in conjunction with the violation of the Bell-Żukowski inequality. The main point of the result is that the Bell-Żukowski operator can be represented by the Bell-Mermin operator.

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I. INTRODUCTION

Bell inequalities that correlation functions satisfying local realistic theories must obey can be violated by certain quantum predictions, as Bell reported in 1964 [1]. Bell used the singlet state, or EPR pairs [2], to show that the correlation functions measured in such singlet states cannot be modeled by local realistic models. Likewise, a certain set of correlation functions produced by quantum measurements of a quantum state contradicts certain predictions of local realistic theories. Those states also cannot be modeled by local realistic models. Up to now, local realistic theories have been studied extensively [3–5]. Many experiments have shown that Bell inequalities and local realistic theories are violated [6–10]. Later, in a work by Fine [11], a set of correlation functions can be described with the property that they are reproducible by local realistic theories for a system in two-partite states if and only if the set of correlation functions satisfies the complete set of (two-setting) Bell inequalities. This result is generalized [12, 13] to a system described by multipartite states in the case where two dichotomic observables are measured per site. In this paper, we present a method using two Bell operators [14] to refute local realistic models of a quantum state. In order to do so, we need only a two-setting and two-particle Bell experiment reproducible by local realistic theories. Such a Bell experiment also reveals the conflict between local realism and quantum mechanics in the sense that the Bell-Żukowski inequality [15] is violated.

Let us consider two-qubit states that, under specific settings, give correlation functions reproducible by local realistic theories. Imagine that N copies of the states can be distributed among $2N$ parties in such a way that each pair of parties shares one copy of the state. The parties perform a Bell-Greenberger-Horne-Zeilinger (GHZ) $2N$-particle experiment [12,13,16,17] on their qubits. Each of the pairs of parties uses the measurement settings noted above. The Bell-Mermin operator [14, 18], B, for their experiment does not show any violation of local realism. Nevertheless, one can find another Bell operator, which differs from B by a numerical factor, that does show such a violation. That is, the original two-qubit states cannot be modeled by local realistic models.

More specifically, the situation is as follows: A given two-setting and two-particle Bell experiment is reproducible by local realistic theories. However, the experimental correlation functions can compute a violation of the Bell-Żukowski inequality. Therefore, actually measured data reveal that the measured state cannot be modeled by local realistic models. Thus, a conflict between local realism and quantum mechanics is revealed. We can see this phenomenon by the simple algebra presented below.

This phenomenon can occur when the system is in a mixed two-qubit state. We analyze the threshold visibility for two-particle interference to reveal the conflict mentioned above. It is found that the threshold visibility agrees with the value to obtain a violation of the Bell-Żukowski inequality.

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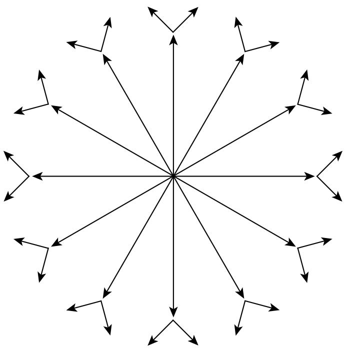

Fig. 1. Schematic diagram of a standard two-setting Bell experiment in an entangled state in twelve particles with the Bell-Mermin operator $B_{\mathbf{N}_{2N}} = 2^{(2N-1)/2}(|\Psi_0^+\rangle\langle\Psi_0^+| - |\Psi_0^-\rangle\langle\Psi_0^-|)$ acting on Greenberger-Horne-Zeilinger states $|\Psi_0^\pm\rangle = (|0^{\otimes 2N}\rangle \pm |1^{\otimes 2N}\rangle)/\sqrt{2}$.

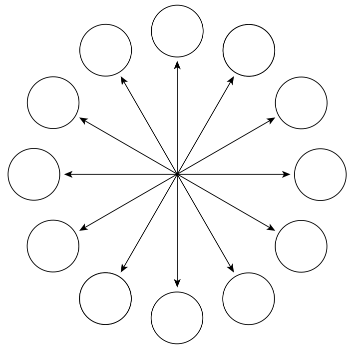

Fig. 2. Schematic diagram of a Bell-Żukowski experiment in an entangled state of twelve particles with the Bell-Żukowski operator $Z_{2N} = \frac{1}{2}\left(\frac{\pi}{2}\right)^{2N}\frac{1}{2^{(2N-1)/2}}B_{\mathbf{N}_{2N}}$.

II. BELL-MERMIN OPERATOR AND BELL-ŻUKOWSKI OPERATOR

Let us consider the following specific Bell-Mermin operator (see Eq. (16)):

$$B_{\mathbf{N}_{2N}} = C_B(|\Psi_0^+\rangle\langle\Psi_0^+| - |\Psi_0^-\rangle\langle\Psi_0^-|), \tag{1}$$

where C_B is a constant and the states $|\Psi_0^\pm\rangle$ are Greenberger-Horne-Zeilinger (GHZ) states [19], *i.e.*,

$$|\Psi_0^\pm\rangle = \frac{1}{\sqrt{2}}(|0^{\otimes 2N}\rangle \pm |1^{\otimes 2N}\rangle). \tag{2}$$

An average of the Bell-Mermin operator, $\langle B_{\mathbf{N}_{2N}}\rangle$, is, then, evaluated by using a standard two-setting Bell experiment. Figure 1 depicts the standard two-setting Bell experiment in an entangled state in twelve particles with the Bell-Mermin operator $B_{\mathbf{N}_{2N(N=6)}}$ acting on the GHZ states $|\Psi_0^\pm\rangle$.

Also, a $2N$-partite Bell operator Z_{2N} [20], which we may call the Bell-Żukowski operator, can be introduced as

$$Z_{2N} = C_Z(|\Psi_0^+\rangle\langle\Psi_0^+| - |\Psi_0^-\rangle\langle\Psi_0^-|), \tag{3}$$

where the numerical coefficient is $C_Z = (\pi/2)^{2N}/2$. An average of the Bell-Żukowski operator is evaluated by using an all-setting Bell experiment, as depicted in Fig. 2, where a Bell-Żukowski experiment is performed on an entangled state of $2N$ particles.

The Bell-Żukowski operator, Z_{2N}, is different from the Bell-Mermin operator, $B_{\mathbf{N}_{2N}}$, given in Eq. (1), only by a numerical factor, C_B/C_Z. Therefore, the specific two-setting Bell $2N$-particle experiment in question computes an average value of the Bell-Żukowski operator, $\langle Z_{2N}\rangle$, when an average value of the Bell-Mermin operator, $\langle B_{\mathbf{N}_{2N}}\rangle$, is evaluated as

$$\langle Z_{2N}\rangle = \frac{C_Z}{C_B}\langle B_{\mathbf{N}_{2N}}\rangle. \tag{4}$$

This argument becomes valid, of course, under the assumption of quantum mechanics, as shown in Fig. 3. In particular, the Bell-Żukowski inequality, $|\langle Z_{2N}\rangle| \leq 1$ [15], is derived under the assumption of predetermined 'hidden' results of the measurement for *all directions in the rotation plane* for the system in a particular state, or *the rotational invariance* of the 'hidden' results. The Bell-Mermin inequality is, however, derived under the assumption of predetermined 'hidden' results of the measurement for *two directions* for the system in the particular state. Therefore the validity of quantum mechanics, in particular, the assumption of rotational invariance is tested in conjunction with the equivalence between the standard two-setting Bell experiment, in an entangled $2N$-particle state and the Bell-Żukowski experiment, or a set of N copies of a two-qubit Bell experiment.

The Bell-Żukowski inequality, $|\langle Z_{2N}\rangle| \leq 1$, imposes an inequality on the average value of the Bell-Mermin operation, $\langle B_{\mathbf{N}_{2N}}\rangle$, *i.e.*,

$$|\langle B_{\mathbf{N}_{2N}}\rangle| \leq \frac{C_B}{C_Z}. \tag{5}$$

We see that a violation of the condition in Eq. (5) implies a violation of the Bell-Żukowski inequality. In this Letter, we compute an expectation value of the Bell-Mermin operator given in Eq. (1) by using a two-particle Bell experiment reproducible by local realistic theories. The Bell-Żukowski inequality is stronger than the standard Bell inequalities for $N \geq 2$. This is why a standard Bell experiment reproducible by local realistic theories reveals the conflict between local realism and quantum mechanics.

III. EXPERIMENTAL SITUATION

We consider that there are $2N$ parties sharing N copies of two-qubit states in such a way that each pair of the N pairs of the parties shares one copy of a two-qubit state. The density matrix of the entangled $2N$-particle state

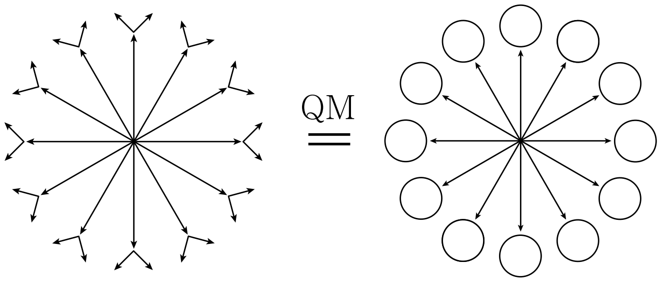

Fig. 3. Schematic diagram of the equivalence between a Bell-Żukowski experiment and a standard two-setting Bell experiment under the validity of quantum mechanics.

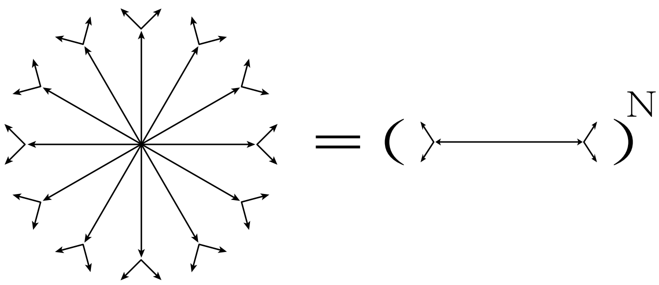

Fig. 4. Schematic diagram of N copies of two-qubit experiments which are equivalent to a standard two-setting Bell experiment in an entangled $2N$-particle state.

becomes

$$\rho_{1,2,\cdots,2N} = \rho_{1,2} \otimes \rho_{3,4} \otimes \cdots \otimes \rho_{2N-1,2N}, \tag{6}$$

where $\rho_{a,b}$ denotes a pair of particles in a two-qubit state, or

$$\rho_{a,b} = V|\psi\rangle\langle\psi| + (1-V)\rho_{\text{noise}} \ (0 \le V \le 1). \tag{7}$$

The value of V is the interferometric visibility of the two-particle correlation experiment for the two-qubit state in a mixed state of a Bell state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+^a;+^b\rangle - i|-^a;-^b\rangle) \tag{8}$$

and the random admixture $\rho_{\text{noise}} = \mathbb{1}/4$.

Then, spatially separated $2N$ observers perform measurements on each of $2N$ particles. If space-like intervals separate the events, those N measurements become equivalent to a standard two-setting Bell experiment in the entangled $2N$-particle state, $\rho_{1,2,\cdots,2N}$, in Eq. (6). The schematic diagram of the equivalence between N copies of two-qubit experiments and a standard two-setting Bell experiment in the entangled $2N$-particle state is shown in Fig. 4.

In the Bell state in Eq. (8), the states $|\pm^k\rangle$ denote eigenstates of the z-component of the Pauli observable, σ_z^k, for the kth observer. Then, for the two-qubit state, $\rho_{a,b}$, we get

$$\begin{aligned}
&\mathrm{tr}[\rho_{a,b}\sigma_x^a\sigma_x^b] = \mathrm{tr}[\rho_{a,b}\sigma_y^a\sigma_y^b] = 0,\\
&\mathrm{tr}[\rho_{a,b}\sigma_x^a\sigma_y^b] = \mathrm{tr}[\rho_{a,b}\sigma_y^a\sigma_x^b] = V,
\end{aligned} \tag{9}$$

where a and b are the labels of two parties and σ_x^k and σ_y^k are the x-component and the y-component of Pauli-spin operators, respectively. It is known (see, for example, in Ref. 11) that the local realistic condition imposes a set of inequalities for the following combinations of the joint probabilities:

$$\begin{aligned}
&|\mathrm{tr}[\rho_{a,b}\sigma_x^a\sigma_x^b] - \mathrm{tr}[\rho_{a,b}\sigma_y^a\sigma_y^b] \pm \mathrm{tr}[\rho_{a,b}\sigma_x^a\sigma_y^b] \pm \mathrm{tr}[\rho_{a,b}\sigma_y^a\sigma_x^b]| = 2V \le 2,\\
&|\mathrm{tr}[\rho_{a,b}\sigma_x^a\sigma_x^b] \mp \mathrm{tr}[\rho_{a,b}\sigma_y^a\sigma_y^b] \pm \mathrm{tr}[\rho_{a,b}\sigma_x^a\sigma_y^b] - \mathrm{tr}[\rho_{a,b}\sigma_y^a\sigma_x^b]| = 0 \le 2.
\end{aligned} \tag{10}$$

We now perform a two-orthogonal-setting Bell-GHZ $2N$-particle correlation experiment with two measurement observables σ_x^k and σ_y^k. These two settings of measurements performed on each pair of parties to evaluate the inequalities in Eq. (10). The inequalities for the correlation measurements on a pair of particles in a two-qubit state is simply applicable to obtain the inequalities for the set of $2N$ particles in N two-qubit states. Therefore, it should be that given 2^{2N} correlation functions are described with the property that they are reproducible by local realistic theories. In other words, the Bell-Mermin operators do not show any violation of local realism as shown below.

The Bell-Mermin operators $B_{\mathbf{N}_{2N}}$ and $B'_{\mathbf{N}_{2N}}$ defined with the measurement observables σ_x^k and σ_y^k are

$$f(B_{\mathbf{N}_{2N}}, B'_{\mathbf{N}_{2N}}) = \prod_{k=1}^{2N} f(\sigma_x^k, \sigma_y^k), \tag{11}$$

where $f(x,y) = \frac{1}{\sqrt{2}}e^{-i\pi/4}(x+iy)$, x, $y \in \mathbf{R}$ is invertible as $x = \Re(f) - \Im(f)$ and $y = \Re(f) + \Im(f)$ [16,18]. Then the Bell-Mermin inequality can be expressed as [18]

$$|\langle B_{\mathbf{N}_{2N}}\rangle| \le 1, \quad |\langle B'_{\mathbf{N}_{2N}}\rangle| \le 1. \tag{12}$$

From the definition of the function f, the measurement settings of σ_x^k and σ_y^k are mapped to $|+^k\rangle\langle -^k|$ and $|-^k\rangle\langle +^k|$ as

$$f(\sigma_x^k, \sigma_y^k) = e^{-i\frac{\pi}{4}}\sqrt{2}|+^k\rangle\langle -^k|, \tag{13}$$

as $\sigma_x^k = |+^k\rangle\langle -^k| + |-^k\rangle\langle +^k|$, $\sigma_y^k = -i|+^k\rangle\langle -^k| + i|-^k\rangle\langle +^k|$ and $f(\sigma_x^k, \sigma_y^k) = e^{-i\pi/4}/\sqrt{2}(\sigma_x^k + i\sigma_y^k)$ [21].

Furthermore, the product of the function of the measurement settings in Eq. (13) for all the pairs of $2N$ particles is obtained as

$$\begin{aligned}\prod_{k=1}^{2N} f(\sigma_x^k, \sigma_y^k) &= e^{-i\frac{2N\pi}{4}} 2^N \prod_{k=1}^{2N} |+^k\rangle\langle -^k| \\ &= e^{-i\frac{2N\pi}{4}} 2^N |+^{\otimes 2N}\rangle\langle -^{\otimes 2N}|. \end{aligned}\tag{14}$$

Therefore, we get the Bell-Mermin operator $B_{\mathbf{N}_{2N}}$

$$\begin{aligned} B_{\mathbf{N}_{2N}} = \; & 2^N\big\{\tfrac{1}{2}(e^{-i\frac{2N\pi}{4}}|+^{\otimes 2N}\rangle\langle -^{\otimes 2N}| + H.c.) \\ & + \tfrac{i}{2}(e^{-i\frac{2N\pi}{4}}|+^{\otimes 2N}\rangle\langle -^{\otimes 2N}| - H.c.)\big\} \\ = \; & \tfrac{2^N}{\sqrt{2}}(e^{-i\frac{(2N-1)\pi}{4}}|+^{\otimes 2N}\rangle\langle -^{\otimes 2N}| + H.c.), \end{aligned}\tag{15}$$

or

$$B_{\mathbf{N}_{2N}} = \frac{2^N}{\sqrt{2}}(|\Psi_0^+\rangle\langle\Psi_0^+| - |\Psi_0^-\rangle\langle\Psi_0^-|). \tag{16}$$

where $e^{-i(2N-1)\pi/4}|+^{\otimes 2N}\rangle = |1^{\otimes 2N}\rangle$. The constant C_B in the Bell-Mermin operator in Eq. (1)is obtained as $C_B = 2^N/\sqrt{2}$. This implies that the 2^{2N} correlation function measured by the Bell-Mermin operator, $B_{\mathbf{N}_{2N}}$, performed on the prepared entangled state of $2N$ particles is represented as a function of N measurements on each pair of $2N$ particles.

Furthermore, for any subset $\alpha \subset \mathbf{N}_{2N}$, B_α and B'_α become

$$f(B_\alpha, B'_\alpha) = \prod_{k\in\alpha} f(\sigma_x^k, \sigma_y^k). \tag{17}$$

When $\alpha, \beta (\subset \mathbf{N}_{2N})$ are disjoint,

$$f(B_{\alpha\cup\beta}, B'_{\alpha\cup\beta}) = f(B_\alpha, B'_\alpha) \otimes f(B_\beta, B'_\beta), \tag{18}$$

and, as a result, we get the following relations,

$$\begin{aligned} B_{\alpha\cup\beta} &= (1/2)B_\alpha \otimes (B_\beta + B'_\beta) \\ &\quad + (1/2)B'_\alpha \otimes (B_\beta - B'_\beta), \\ B'_{\alpha\cup\beta} &= (1/2)B'_\alpha \otimes (B'_\beta + B_\beta) \\ &\quad + (1/2)B_\alpha \otimes (B'_\beta - B_\beta). \end{aligned}\tag{19}$$

Therefore, we obtain the average value of the Bell-Mermin operator as a simple function of the visibility of the two-particle correlation function

$$\langle B_{\mathbf{N}_{2N}}\rangle = \prod_{i=2}^{N}\langle B_{\{i-1,i\}}\rangle = V^N. \tag{20}$$

This says that the average of the Bell-Mermin operator, $B_{\mathbf{N}_{2N}}$, performed on the $2N$ particles in the entangled N-qubit states is the product of N measurements on each pair of $2N$ particles.

IV. CONCLUSION

As a result of the Bell-Żukowski inequality and the relation between the Bell-Żukowski operator and the Bell-Mermin operator, $C_B/C_Z \leq 1$ and $V^N < 1$. Therefore we have a violation of the condition in Eq. (5) for the value of the visibility in the range of

$$\sqrt{2}\left(\frac{8}{\pi^2}\right)^N < V \leq 1, \tag{21}$$

for $N > 2$ as $C_B = 2^N/\sqrt{2}$ and $C_Z = (\pi/2)^{2N}/2$. So, for the given value of V, the violation of the measured two-qubit state cannot be modeled by a local realistic theory.

We have shown that the Bell-Żukowski operator can be represented by the Bell-Mermin operator. This fact provides a means to check *whether a quantum state can be modeled by local realistic models*, *i.e.*, if the conflict between local realism and quantum mechanics occurs. We have used only a two-setting and two-particle Bell experiment that is reproducible by local realistic theories. Given a two-setting and two-particle Bell experiment reproducible by local realistic theories, one can compute a violation of the Bell-Żukowski inequality. Measured data, thus, indicate that the measured state cannot be modeled by local realistic models and, as a result, the conflict between local realism and quantum mechanics is revealed.

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